



A characterization of signed graphs with generalized perfect elimination orderings

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ABSTRACT

An important property of chordal graphs is that these graphs are characterized by the existence of perfect elimination orderings on their vertex sets. In this paper, we generalize the notion of perfect elimination orderings to signed graphs, and give a characterization for graphs admitting such orderings, together with characterizations restricted to some subclasses and further properties of those graphs. The definition of our generalized perfect elimination orderings is motivated by a generalization of the classical result that a so-called graphic hyperplane arrangement is free if and only if the corresponding graph is chordal.

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1. Introduction

1.1. Background and summary of results

An undirected graph is called *chordal* if any cycle with at least four vertices has a *chord* (an edge not in the cycle with both endpoints in the cycle). Chordal graphs are a classical subject in the graph theory and these graphs have been playing significant roles also in several related research areas. A property frequently used in such research is that a graph is chordal if and only if it admits a special ordering of vertices, called a *perfect elimination ordering* or a *vertex elimination ordering* (see [7, Section 7]). Roughly speaking, perfect elimination orderings correspond to a kind of growing process from an empty graph to the given graph, in which a new vertex is pasted to the present graph at a clique. This characterization of chordal graphs is very significant, since it connects combinatorial properties of the graph to geometric ones. For example, a famous result regarding hyperplane arrangements, given by Edelman and Reiner [6, Theorem 3.3] (by virtue of the key result by Stanley [11, Proposition 2.8]), states that an arrangement parameterized by a graph in a certain manner is “free” if and only if the corresponding graph is chordal. The present work is motivated by a generalization of this equivalence (see Section 1.2).

The aim of this paper is to generalize the notion of perfect elimination orderings (and even the notion of chordal graphs) to signed graphs, i.e. graphs with each edge having a sign “+” or “−”, and to give a complete characterization of a signed graph admitting such an ordering. In this paper we call such an ordering and such a graph a *signed-elimination ordering* and a *signed-eliminable graph*, respectively. A signed-elimination ordering is such that it is a usual perfect elimination ordering when restricted to edges with a fixed sign, and it satisfies a further condition across the two signs (see Definition 3.1 for a precise definition). Then our characterization (Theorem 5.1) says that a signed graph is signed-eliminable if and only if the subgraph restricted to each sign is chordal and it satisfies certain further conditions involving edges with both signs. The characterization implies that it is indeed a generalization of the aforementioned classical equivalence of chordality to admitting perfect elimination orderings.

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1.2. Related works

As mentioned in Section 1.1, the present work is motivated by a result of Edelman and Reiner [6, Theorem 3.3] on a relation of graphs and hyperplane arrangements. More precisely, it was shown by [6, Theorem 3.3] that for a sub-arrangement \mathcal{A} of Coxeter arrangement of type A_n parameterized by a graph G , chordality of G (equivalently, existence of perfect elimination ordering on G) is equivalent to freeness of \mathcal{A} as well as to supersolvability of \mathcal{A} . (Note that supersolvability generally implies freeness; see [9, Theorem 4.3.21].) There has been an early generalization of perfect elimination orderings to signed graphs, given by Zaslavsky [13], which aims at generalizing the study of sub-arrangements of Coxeter arrangements of type A_n to other types such as type B_n . He showed [12] that for the case of type B_n , a sub-arrangement is supersolvable if and only if the corresponding signed graph admits his generalized perfect elimination ordering, thus extending a part of Edelman and Reiner's result to signed graphs. (Note that for the case of type B_n , Edelman and Reiner [6] also gave characterizations of free sub-arrangements and supersolvable ones in terms of graphs that are not signed but have loops; and that Cordovil, Forge and Klein [4] also studied a relation between chordality and supersolvability.)

On the other hand, in contrast to Zaslavsky's work that focused on the supersolvability, the motivation of the present work focuses on the equivalence in Edelman and Reiner's result between *freeness* and existence of perfect elimination ordering. Recently, Abe, Numata and the author [2] would like to extend this equivalence to Coxeter arrangements of type A_n that are endowed with multiplicities of some types. Our multiplicities are naturally parameterized by signed graphs in such a way that pairs of non-adjacent vertices correspond to “neutral” multiplicities $2m$, and plus- and minus-signed edges correspond to $2m + 1$ and $2m - 1$, respectively (the case $m = 0$ corresponds to the original case of sub-arrangements of type A_n). We introduced another generalization (studied in the present paper) of perfect elimination orderings to signed graphs, and proved first that if a signed graph admits a signed-elimination ordering (in our sense) then the corresponding multiarrangement is free. Then we proved the reverse implication by using the characterization of existence of signed-elimination ordering given in the present paper. Thus we extended a part of Edelman and Reiner's result to the case of multiarrangements by virtue of the result of the present paper. See [2] for more details.

In comparison to the above-mentioned work of Zaslavsky, one may feel the lack of results on relations to supersolvability as a drawback of our generalization of perfect elimination orderings. However, it would be simply differences of scopes and directions; as Zaslavsky's work focused on supersolvability but not on freeness and extended it from arrangements of type A_n to type B_n ; while our work focused on freeness but not on supersolvability and extended it from simple arrangements to multiarrangements. (In fact, to the author's best knowledge the notion of supersolvability seems not yet extended to the case of multiarrangements.) Owing to the clear differences, it would not be curious that there seems no implication from one of Zaslavsky's and our generalized perfect elimination orderings to the other. It is also worthy to notice that Zaslavsky's work did not present a generalization of *chordality* to signed graphs (see [12, Section 5.2]), while our characterization (Theorem 5.1) shows a generalization of chordality that is compatible to the generalized perfect elimination orderings. Moreover, our above-mentioned work [2] that uses the result of the present paper is also relevant to a conjecture proposed by Athanasiadis [3] on the freeness characterization for another class of hyperplane arrangements; one direction of the conjecture (namely, implication of freeness from the conditions proposed by Athanasiadis) is proven in [2, Section 5] as an application of our results. These would show the significance of the generalization of perfect elimination orderings proposed in the present paper.

Finally, note that a recent work by Mckee [8] also generalized the notion of chordal graphs to signed graphs. However, his generalization was done in a very different manner and motivation from ours, and there seems no obvious relation between his and our generalizations.

1.3. Organization of the paper

This paper is organized as follows. In Section 2, we present and fix notations and terminology for graphs and for signed graphs, and also give some lemmas for later references. In Section 3, we introduce the notion of signed-elimination orderings and signed-eliminable graphs, state and prove some fundamental properties, and also give a greedy algorithm for deciding whether a given graph is signed-eliminable and constructing a signed-elimination ordering (if it exists). Section 4 is an introduction to the full characterization of signed-eliminable graphs; we give definitions of two kinds of exceptional subgraphs (called *mountains* and *hills*), prove that any signed graph with three vertices is signed-eliminable, and present some further properties. Section 5 is devoted to the statement and the proof of our full characterization. Finally, in Section 6, we give characterizations of the signed-eliminable graphs in some subclasses (graphs with four vertices; chordal graphs; graphs with independence number less than three; and complete graphs) by restricting our full characterization to these subclasses.

2. Preliminaries

2.1. Graphs

In this paper every graph $G = (V, E)$ is finite, simple and undirected. See any textbook of graph theory, e.g. [5], for basic notations and terminology. We denote $v \text{---} w$ and $v \not\text{---} w$, respectively, to signify that $vw \in E$ and $vw \notin E$, where vw

denotes the unordered pair of v and w . For $V' \subset V$, let $G|_{V'}$ denote the induced subgraph of G with vertex set V' , and write $G \setminus V' = G|_{V \setminus V'}$. In this paper, we often abbreviate a singleton $\{x\}$ simply to x unless some ambiguity arises. For $v \in V$, we write

$$N_G(v) = \{w \in V | v \text{---} w\} \quad \text{and} \quad N_G[v] = N_G(v) \cup v,$$

and for $V_1, V_2 \subset V$, define

$$[V_1, V_2] = \{vw | v \in V_1, w \in V_2, v \neq w\}$$

(note that we do *not* assume that $[V_1, V_2] \subset E$). We write $N(v) = N_G(v)$ and $N[v] = N_G[v]$ if the graph G is obvious from the context. A graph G is called *chordal* if it has no induced cycle of length at least four. We refer to a bijection ν from V to $\{1, 2, \dots, |V|\}$ as an *ordering* on G . The following theorem is a well-known characterization of chordal graphs:

Theorem 2.1 (See e.g. [7]). *A graph G is chordal if and only if there is an ordering ν on G such that for any three vertices u, v and w of G with $\nu(u) < \nu(w) < \nu(v)$, if $u \text{---} w \text{---} v$ then $u \text{---} v$.*

An ordering ν satisfying the condition in this theorem is called a *perfect elimination ordering*, or simply an *elimination ordering*. Now a straightforward argument shows the following properties:

Lemma 2.2. *Let G be a graph and $v \in V$.*

- (1) *If ν is an elimination ordering on G with $\nu(v) = |V|$, then $N_G[v]$ is a clique of G and the restriction of ν on $V \setminus v$ is also an elimination ordering on $G \setminus v$.*
- (2) *Conversely, suppose that $N_G[v]$ is a clique of G and ν is an elimination ordering on $G \setminus v$. Then any ordering $\bar{\nu}$ on G which extends ν and satisfies $\bar{\nu}(v) = |V|$ is also an elimination ordering on G .*

We also prepare the following lemma on chordal graphs:

Lemma 2.3. *Let G be a chordal graph and $V' \subsetneq V$ a clique of G . Then there is a vertex $v \in V \setminus V'$ such that $N_G[v]$ is a clique of G .*

Proof. First, an elimination ordering μ on G exists by Theorem 2.1. Let $w = \mu^{-1}(|V|) \in V$. Then $N_G[w]$ is a clique by Lemma 2.2(1). Our claim holds if $w \notin V'$; thus suppose that $w \in V'$. If $N_G[w] = V$ (i.e. G is a complete graph), then any vertex in $V \setminus V'$ satisfies the claim. On the other hand, suppose that $N_G[w] \neq V$. Then we have $N_G(w) \subsetneq V \setminus w$ and $N_G(w)$ is a clique of $G \setminus w$, therefore induction on $|V|$ enables us to take a vertex $v \in (V \setminus w) \setminus N_G(w) = V \setminus N_G[w]$ such that $N_{G \setminus w}[v]$ is a clique in $G \setminus w$. Moreover, we have $v \not\text{---} w$ by the choice of v , therefore $N_G[v] = N_{G \setminus w}[v]$ is also a clique in G . Hence the claim holds, since $V' \subset N_G[w]$. \square

2.2. Signed graphs

A *signed graph* is a graph $G = (V, E)$ with a partition $E = E_+ \cup E_-$ of edge set (where each part may be empty). For $\sigma \in \{+, -\}$, we write $G_\sigma = (V, E_\sigma)$, and denote $v \xrightarrow{\sigma} w$ and $v \not\xrightarrow{\sigma} w$, respectively, to signify that $vw \in E_\sigma$ and $vw \notin E_\sigma$. We simply write $N_{G_\sigma}[v] = N_\sigma[v]$ and $N_{G_\sigma}(v) = N_\sigma(v)$ if the underlying graph G is obvious from the context. In this paper, single and double edges in a figure of a graph represent edges with different signs.

The following simple lemma will be used in our argument later:

Lemma 2.4. *Let $G = (V, E)$ be a connected signed graph with $E_+ \neq \emptyset$ and $E_- \neq \emptyset$. Then we have $v \overset{+}{\text{---}} v' \overset{-}{\text{---}} v''$ for some $v, v', v'' \in V$.*

Proof. By the assumption, vertex sets of an edge in E_+ and of an edge in E_- are joined by a path. This implies that G involves a path $x_1 x_2 \cdots x_k$ with $x_1 \overset{+}{\text{---}} x_2$ and $x_{k-1} \overset{-}{\text{---}} x_k$. Now this path must involve a desired triple. \square

3. Generalization of elimination orderings to signed graphs

3.1. Definition

As a generalization of perfect elimination orderings for non-signed graphs to signed graphs, here we introduce the following notion:

Definition 3.1. Let $G = (V, E)$ be a signed graph and ν an ordering on G . Then we say that ν is a *signed-elimination ordering*, or a *SEO* in short, if for any triple (u, v, w) of vertices of G such that $\nu(u) < \nu(w) < \nu(v)$, and for each $\sigma \in \{+, -\}$, we have

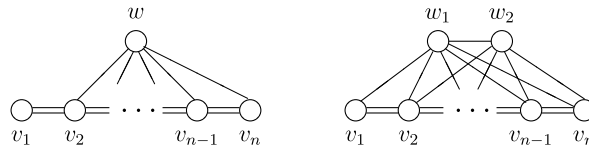


Fig. 1. Examples of SE graphs.

(E1) if $u \xrightarrow{\sigma} w \xrightarrow{\sigma} v$, then $u \xrightarrow{\sigma} v$;

(E2) if $u \xrightarrow{\sigma} v \xrightarrow{-\sigma} w$, then $u \xrightarrow{\sigma} w$.

We call the graph G *signed-eliminable*, or *SE* in short, if a SEO on G exists.

In other words, when we assign weights $\omega(vv')$ to pairs vv' of vertices v, v' of G by the rule that $\omega(vv') = \pm 1$ and 0 if $vv' \in E_{\pm}$ and $vv' \notin E$, respectively, it follows that SEOs are the orderings ν such that for any triple (u, v, w) with $\nu(u) < \nu(v) < \nu(w)$, if $a \leq b \leq c$ are three weights $\omega(uv)$, $\omega(vw)$ and $\omega(uw)$ in nondecreasing order, then $b = \omega(uv)$ unless $u \not\sim w$. This definition is motivated by recent research on hyperplane arrangements by Abe, Nuida and Numata (see [2]), that generalize Edelman and Reiner's characterization [6, Theorem 3.3] of certain “free” arrangements in terms of existence of a perfect elimination ordering on the corresponding graph.

Example 3.2. The signed graphs in Fig. 1 are signed-eliminable for any $n \geq 2$. For the graph on the left, a SEO is given by $w \mapsto 1$ and $v_i \mapsto i + 1$. On the other hand, for the graph on the right, a SEO is given by $w_1 \mapsto 1$, $w_2 \mapsto 2$ and $v_i \mapsto i + 2$.

Remark 3.3. By condition (E1), a SEO on G is also a perfect elimination ordering on both G_+ and G_- , thus Theorem 2.1 implies that G_+ and G_- must be chordal if G is SE. In particular, when $E_+ = \emptyset$ or $E_- = \emptyset$, the SEOs on G are precisely the perfect elimination orderings on G , therefore in this case G is SE if and only if G is chordal. Thus SEOs are a generalization of the usual perfect elimination orderings.

Remark 3.4. The restriction of any SEO on a signed graph to its induced subgraph is also a SEO. Thus the property of being SE is closed under taking induced subgraphs.

Remark 3.5. A signed graph is SE if and only if every connected component of the graph is SE.

The aim of this paper is to give a characterization of SE graphs.

3.2. Fundamental properties

In this subsection, we present fundamental properties of SE graphs for later references. Let $G = (V, E)$ be a signed graph. We start with the following observation:

Lemma 3.6. Suppose that ν is a SEO on G , and $\nu = \nu^{-1}(|V|) \in V$. Then the restriction $\nu|_{V \setminus \nu}$ of ν is a SEO on $G \setminus \nu$, and the following conditions hold:

(S1) For each $\sigma \in \{+, -\}$, $N_{G_{\sigma}}[\nu]$ is a clique in G_{σ} (that is, ν is simplicial in G_{σ}).

(S2) For each $\sigma \in \{+, -\}$, if $u \xrightarrow{-\sigma} w \xrightarrow{\sigma} \nu$, then $u \xrightarrow{-\sigma} \nu$.

We call a vertex $\nu \in V$ *signed-simplicial* if it satisfies these two conditions.

Proof. The first claim follows from Remark 3.4. For the second claim, the first condition is satisfied by Lemma 2.2 and Remark 3.3. For the second condition, since $\nu(u) < \nu(v)$ and $\nu(w) < \nu(v)$, we have $u \xrightarrow{-\sigma} \nu$ by condition (E2) in Definition 3.1. Hence the claim holds. \square

Remark 3.7. If $\nu \in V$ is signed-simplicial, then $N_{\sigma}[\nu]$ is a maximal clique of G_{σ} for each $\sigma \in \{+, -\}$. On the other hand, if $\nu \in V$ and $N_G[\nu] = V$, then condition (S1) for ν implies condition (S2) for ν .

Owing to Lemma 3.6, we introduce the following notation:

Definition 3.8. Let $S(G)$ denote the set of the signed-simplicial vertices of G ; thus $S(G) \neq \emptyset$ if G is signed-eliminable.

Remark 3.9. By the definition of $S(G)$, we have $\nu \in S(G|_{V'})$ if $\nu \in S(G)$ and $\nu \in V' \subset V$.

Our next result shows that the “converse” of Lemma 3.6 is also valid:

Lemma 3.10. Suppose that $\nu \in S(G)$ and ν is a SEO on $G \setminus \nu$. Then the unique extension $\bar{\nu}$ of ν to V with $\bar{\nu}(\nu) = |V|$ is also a SEO on G . Any SEO on G is obtained in such a manner.

Proof. First, note that the last claim is a restatement of Lemma 3.6. To prove that \bar{v} is a SEO, since v is a SEO on $G \setminus v$, it suffices to show that conditions (E1) and (E2) are satisfied for \bar{v} when v plays the role of w in these conditions. Now (E1) and (E2) follow from the conditions (S1) and (S2), respectively, for v to be signed-simplicial. \square

Corollary 3.11. Suppose that G is signed-eliminable. Then for any $v \in S(G)$, there is a SEO ν on G such that $\nu(v) = |V|$.

Proof. Remark 3.4 implies that $G \setminus v$ is SE, therefore a SEO ν on $G \setminus v$ exists. This ν extends to the desired ordering on G by Lemma 3.10. \square

3.3. An algorithm to find signed-elimination orderings

Summarizing the results in the previous sections, here we give a greedy algorithm which enables us to decide whether or not a given signed graph G is signed-eliminable and to construct a SEO on G (whenever it exists). The next lemma is a key ingredient of our algorithm:

Lemma 3.12. Let $G = (V, E)$ be a signed graph.

(1) Let ν be a SEO on G , and put $v_i = \nu^{-1}(i) \in V$ for $1 \leq i \leq |V|$. Then

$$v_i \in S(G|_{\{v_1, v_2, \dots, v_i\}}) \quad \text{for each } 1 \leq i \leq |V|. \quad (1)$$

(2) Conversely, let $V = \{v_1, v_2, \dots, v_n\}$ be a numbering of elements of V satisfying the condition (1). Then the map $\nu : v_i \mapsto i$ is a SEO on G .

Proof. Put $V_i = \{v_1, \dots, v_i\}$. The former claim follows from Remark 3.4 and Lemma 3.6; namely, $\nu|_{V_i}$ is a SEO on $G|_{V_i}$ for each i . On the other hand, for the latter claim, it follows from Lemma 3.10 and induction on i that the restriction of ν to V_i is a SEO on $G|_{V_i}$. Thus the claim holds since $V_n = V$. \square

Now our algorithm is described as follows:

Theorem 3.13. Consider the following algorithm (with input G):

Step 1: If $V = \emptyset$, then output an empty sequence $()$. Otherwise, go to Step 2.

Step 2: Find a vertex $v \in S(G)$ (by, for example, checking the condition of being signed-simplicial for every vertex) and go to Step 3. If such a vertex does not exist, output NULL.

Step 3: Perform this algorithm recursively for input $G \setminus v$. If it outputs a sequence (w_1, \dots, w_k) , then output a sequence (w_1, \dots, w_k, v) . If it outputs NULL, then output NULL.

Then G is signed-eliminable if and only if the algorithm outputs a (possibly empty) sequence, not NULL. Moreover, if the output is a sequence (v_1, \dots, v_n) , then the map $\nu : v_i \mapsto i$ is a SEO on G .

Proof. First, if the algorithm outputs a sequence (v_1, \dots, v_n) , then this sequence satisfies the condition in Lemma 3.12, (1) by the construction, therefore G is SE with a SEO ν . On the other hand, if G is SE, then a $v \in S(G)$ is found in Step 2 by Lemma 3.6(2), while $G \setminus v$ is also SE by Remark 3.4. Thus the output in Step 3 for input $G \setminus v$ is not NULL by induction on $|V|$, therefore the output for input G is also not NULL. Hence the proof is concluded. \square

3.4. Invariants for signed-eliminable graphs

In this subsection, we introduce the following object associated to each SE graph that can be computed from a given SEO, and prove that it is in fact independent of the choice of the SEO; therefore the object is an invariant for SE graphs. The definition is the following:

Definition 3.14. Let $G = (V, E)$ be a signed-eliminable graph with n vertices and ν a SEO on G . Then for each $1 \leq i \leq n$, define a pair $d^{(\nu)}(i) = (d_+^{(\nu)}(i), d_-^{(\nu)}(i))$ of nonnegative integers by

$$d_\sigma^{(\nu)}(i) = |\{v \in V \mid \nu(v) \leq i \text{ and } v_i \overset{\sigma}{\sim} v\}| \quad \text{for each } \sigma \in \{+, -\},$$

where $v_i = \nu^{-1}(i) \in V$. Moreover, let $d^{(\nu)}$ denote the multiset consisting of all pairs $d^{(\nu)}(i)$ with $1 \leq i \leq n$.

For example, if G is a graph with $v_1 \overset{+}{\sim} v_2 \overset{+}{\sim} v_4$ and $v_3 \overset{-}{\sim} v_4$ (and having no other vertices and no other edges) and ν is a SEO on G such that $\nu(v_i) = i$, then $d^{(\nu)} = \{d^{(\nu)}(1), \dots, d^{(\nu)}(4)\} = \{(0, 0), (1, 0), (0, 0), (1, 1)\}$. For this object, we have the following property:

Proposition 3.15. For any signed-eliminable graph G , the multiset $d^{(\nu)}$ does not depend on the choice of a SEO ν on G . Hence $d^{(\nu)}$ gives an invariant for signed-eliminable graphs.

Proof. Let ν and μ be two SEOs on the same G , and put $n = |V|$, $v_i = \nu^{-1}(i) \in V$, $w_i = \mu^{-1}(i) \in V$, $V_i = \{v_1, \dots, v_i\}$ and $G_i = G|_{V_i}$. First, we show that $d^{(\nu)} = d^{(\mu)}$ (as multisets) if $v_{i-1} = w_i$ and $v_i = w_{i-1}$ for some $2 \leq i \leq n$ and $v_j = w_j$ for any $1 \leq j \leq n$ other than $i-1$ and i . Now we have $d^{(\nu)}(j) = d^{(\mu)}(j)$ for any $1 \leq j \leq n$ other than $i-1$ and i by definition, therefore it suffices to show that either $(d^{(\nu)}(i-1), d^{(\nu)}(i)) = (d^{(\mu)}(i-1), d^{(\mu)}(i))$ or $(d^{(\nu)}(i-1), d^{(\nu)}(i)) = (d^{(\mu)}(i), d^{(\mu)}(i-1))$ holds. If $v_{i-1} \not\sim v_i$, then it follows immediately from the definition of $d^{(\nu)}$ that $d^{(\nu)}(i-1) = d^{(\mu)}(i)$ and $d^{(\nu)}(i) = d^{(\mu)}(i-1)$. On the other hand, suppose that $v_{i-1} \stackrel{\sigma}{\sim} v_i$ for some $\sigma \in \{+, -\}$. Put $X_v^\tau = \{v_j \mid 1 \leq j \leq i-2, v \stackrel{\tau}{\sim} v_j\}$ for $v \in \{v_{i-1}, v_i\}$ and $\tau \in \{+, -\}$. Now we have $X_{v_i}^\sigma \subset X_{v_{i-1}}^\sigma$ by the condition (E1) for G_i and $v|_{V_i}$. Similarly, we have $X_{v_{i-1}}^{-\sigma} \subset X_{v_i}^{-\sigma}$ by the condition (E2) for G_i and $v|_{V_i}$. Moreover, we also have $X_{v_{i-1}}^\sigma \subset X_{v_i}^\sigma$ and $X_{v_i}^{-\sigma} \subset X_{v_{i-1}}^{-\sigma}$ by exchanging the roles of ν and μ (recall that $w_{i-1} = v_i$ and $w_i = v_{i-1}$); thus $X_{v_{i-1}}^{\pm\sigma} = X_{v_i}^{\pm\sigma}$, respectively. This implies that

$$d_{\pm\sigma}^{(\nu)}(i-1) = |X_{v_{i-1}}^{\pm\sigma}| = |X_{v_i}^{\pm\sigma}| = d_{\pm\sigma}^{(\mu)}(i-1),$$

$$d_{-\sigma}^{(\nu)}(i) = |X_{v_i}^{-\sigma}| = |X_{w_i}^{-\sigma}| = d_{-\sigma}^{(\mu)}(i),$$

$$d_{\sigma}^{(\nu)}(i) = |X_{v_i}^{\sigma}| + 1 = |X_{w_i}^{\sigma}| + 1 = d_{\sigma}^{(\mu)}(i),$$

therefore the claim of this paragraph follows.

To conclude the proof, choose the index i with $v_i = w_n$. Now if $i < n$, then we have $w_n \in S(G)$ and $v_{i+1} \in S(G_{i+1})$ by Lemma 3.12(1); therefore $v_i = w_n \in S(G_{i+1})$ and $v_{i+1} \in S(G_{i+1} \setminus v_i)$ by Remark 3.9. By Lemma 3.12, it follows that the ordering ν' on G with $\nu'(v_i) = i+1$, $\nu'(v_{i+1}) = i$ and $\nu'(v_j) = j$ for any $1 \leq j \leq n$ other than i and $i+1$ is also a SEO on G ; therefore $d^{(\nu)} = d^{(\nu')}$ by the previous paragraph. Iterating this process, we obtain a SEO ν'' on G such that $d^{(\nu)} = d^{(\nu')}$ and $\nu''(w_n) = n$; while it follows from induction on n that $d^{(\nu'')|_{V'}} = d^{(\mu)|_{V'}}$ where $V' = V \setminus w_n$, therefore $d^{(\nu'')} = d^{(\mu)}$. Hence we have $d^{(\nu)} = d^{(\mu)}$, concluding the proof. \square

In the special case of non-signed graphs, Proposition 3.15 coincides with a result of Rose [10, Theorem 4]. Note that $d^{(\nu)}(1) = (0, 0)$ for any case. This proposition implies that for any map f , the multiset consisting of $f(d^{(\nu)}(i))$ for all $1 \leq i \leq |V|$ is also an invariant for SE graphs. In particular, we have the following corollary, that plays a significant role in [2]:

Corollary 3.16. For a signed-eliminable graph G , define a multiset $\widetilde{\deg}(G)$ as consisting of the values $d_+^{(\nu)}(i) - d_-^{(\nu)}(i)$ for all $1 \leq i \leq |V|$, where ν is a SEO on G . Then $\widetilde{\deg}(G)$ is independent of the SEO ν ; therefore it is an invariant for signed-eliminable graphs.

4. Lemmas for characterization of signed-eliminable graphs

In this section, we prove that any signed graph with at most three vertices is SE, we present special examples of signed graphs that are not SE, and we give some further auxiliary properties. Let $G = (V, E)$ denote a signed graph throughout this section.

First, we have the following:

Proposition 4.1. If $|V| \leq 3$, then G is always signed-eliminable.

Proof. This is trivial if $|V| \leq 2$. For the case $|V| = 3$, Remark 3.3 implies that G is SE if $E_+ = \emptyset$ or $E_- = \emptyset$. On the other hand, if $E_+ \neq \emptyset$ and $E_- \neq \emptyset$, then $N_+(v) \neq \emptyset$ and $N_-(v) \neq \emptyset$ for some $v \in V$, and now we have $v \in S(G)$. Thus Lemma 3.10 and induction on $|V|$ imply that G is SE. Hence the proof is concluded. \square

Secondly, we give the following observations which will be used in our argument several times:

Lemma 4.2. Suppose that V is a disjoint union of V' and V'' , and $v \in S(G|_{V'})$. Suppose further that $\sigma \in \{+, -\}$, $[v, V''] \subset E_\sigma$, V'' is a clique in G_σ , and $[V', V''] \cap E_{-\sigma} = \emptyset$. Then $v \in S(G)$ if the following condition is satisfied:

(D) If $v \stackrel{\sigma}{\sim} w \in V'$ and $w' \in V''$, then $w \stackrel{\sigma}{\sim} w'$.

Proof. Put $G' = G|_{V'}$. For condition (S1), the assumption implies that $N_{G_{-\sigma}}[v] = N_{G'_{-\sigma}}[v]$ is a clique in $G_{-\sigma}$, $N_{G_\sigma}[v] = N_{G'_\sigma}[v] \cup V''$, and the latter set is a clique in G_σ by (D). For condition (S2), suppose that $\tau \in \{+, -\}$ and $u \stackrel{\tau}{\sim} w \stackrel{\tau}{\sim} v$. It suffices to show that $u \stackrel{\tau}{\sim} v$ when $\{u, w\} \not\subset V'$. Now we have $w \in V'$; otherwise, $\tau = \sigma$ but $N_{G_{-\sigma}}(w) = \emptyset$, a contradiction.

This implies that $u \in V''$, $\tau = -\sigma$ (since $u \not\stackrel{\sigma}{\sim} w$) and $u \stackrel{\tau}{\sim} v$. Hence the claim holds. \square

Lemma 4.3. Suppose that $v' \in V$ and $v \in S(G \setminus v')$.

(1) Suppose further that $\sigma \in \{+, -\}$ and $v \stackrel{\sigma}{\sim} v'$. Then $v \in S(G)$ if the following two conditions are satisfied:

(D'1) If $v \stackrel{\sigma}{\sim} w \neq v'$, then $w \stackrel{\sigma}{\sim} v'$.

(D'2) If $v' \stackrel{-\sigma}{\sim} w$, then $v \sim w$.

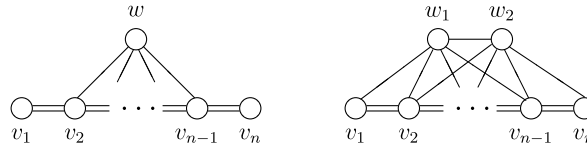


Fig. 2. Examples of non-SE graphs.

(2) Suppose further that $v \not\sim v'$. Then $v \in S(G)$ if the following condition is satisfied:

(D'') $N_{G_\tau}(v) \cap N_{G_{-\tau}}(v') = \emptyset$ for each $\tau \in \{+, -\}$.

Proof. Put $G' = G \setminus v'$. For the former claim, since $v \in S(G')$, condition (D'1) implies that $N_{G_{-\sigma}}[v] = N_{G'_{-\sigma}}[v]$ and $N_{G_\sigma}[v] = N_{G'_\sigma}[v] \cup v'$ are cliques in $G_{-\sigma}$ and G_σ , respectively. Moreover, if $\tau \in \{+, -\}$ and $u \stackrel{\tau}{\sim} v' \stackrel{\tau}{\sim} v$, then $\tau = \sigma$ and $u \stackrel{\sigma}{\sim} v$ by (D'1) and (D'2). If $v' \stackrel{-\tau}{\sim} u \stackrel{\tau}{\sim} v$, then $\tau \neq \sigma$ by (D'1), thus $\tau = -\sigma$ and $v' \stackrel{-\tau}{\sim} v$. Since $v \in S(G')$, these imply that condition (S2) is satisfied. Thus we have $v \in S(G)$.

For the latter claim, we have $N_{G_+}[v] = N_{G'_+}[v]$ and $N_{G_-}[v] = N_{G'_-}[v]$, therefore condition (S1) is satisfied since $v \in S(G')$. On the other hand, since $v \not\sim v'$, (D'') implies that if $u \stackrel{-\sigma}{\sim} w \stackrel{\sigma}{\sim} v$, then $u, w \neq v'$. Since $v \in S(G')$, this implies that condition (S2) is also satisfied. Thus we have $v \in S(G)$. Hence the proof is concluded. \square

Here we introduce the following special signed graphs that are not SE; these graphs will play a significant role in our characterization (compare these graphs with those in Fig. 1):

Definition 4.4. (1) We say that a graph with vertices $v_1, v_2, \dots, v_n; w$ with $n \geq 3$ is a (σ) -mountain, where $\sigma \in \{+, -\}$, if $v_i \stackrel{-\sigma}{\sim} v_{i+1}$ for $1 \leq i \leq n-1$, $w \stackrel{\sigma}{\sim} v_i$ for $2 \leq i \leq n-1$, and any other pair of vertices is not joined by an edge (see the left-hand side of Fig. 2).
 (2) We say that a graph with vertices $v_1, v_2, \dots, v_n; w_1, w_2$ with $n \geq 2$ is a (σ) -hill, where $\sigma \in \{+, -\}$, if $v_i \stackrel{-\sigma}{\sim} v_{i+1}$ for $1 \leq i \leq n-1$, $w_1 \stackrel{\sigma}{\sim} w_2$, $w_1 \stackrel{\sigma}{\sim} v_i$ for $1 \leq i \leq n-1$, $w_2 \stackrel{\sigma}{\sim} v_i$ for $2 \leq i \leq n$, and any other pair of vertices is not joined by an edge (see the right-hand side of Fig. 2).

Lemma 4.5. Any mountain and any hill are not signed-eliminable.

Proof. Let G be a σ -mountain or a σ -hill in Fig. 2 for $\sigma \in \{+, -\}$. Then it suffices to show that $S(G) = \emptyset$ (see Definition 3.8). Now for the case of σ -mountain, $N_{-\sigma}[v_i]$ with $2 \leq i \leq n-1$ is not a clique in $G_{-\sigma}$, while none of w, v_1 and v_n satisfies condition (S2) (focus on the subgraphs $v_1 \stackrel{-\sigma}{\sim} v_2 \stackrel{\sigma}{\sim} w$ and $v_n \stackrel{-\sigma}{\sim} v_{n-1} \stackrel{\sigma}{\sim} w$). On the other hand, for the case of σ -hill, $N_\sigma[w_i]$ and $N_{-\sigma}[v_j]$ are not cliques in G_σ and $G_{-\sigma}$, respectively, for $1 \leq i \leq 2$ and $2 \leq j \leq n-1$, while neither v_1 nor v_n satisfies condition (S2) (focus on the subgraphs $w_2 \stackrel{\sigma}{\sim} v_2 \stackrel{-\sigma}{\sim} v_1$ and $w_1 \stackrel{\sigma}{\sim} v_{n-1} \stackrel{-\sigma}{\sim} v_n$). Thus we have $S(G) = \emptyset$ in both cases. \square

Moreover, we present a key lemma in our argument:

Lemma 4.6. Suppose that G is signed-eliminable and u, v, w and x are distinct vertices of G . If $\sigma \in \{+, -\}$ and $u \stackrel{\sigma}{\sim} v \stackrel{-\sigma}{\sim} w \stackrel{\sigma}{\sim} x$, then $u \stackrel{\sigma}{\sim} x$, and we have either $u \stackrel{\sigma}{\sim} w$ or $v \stackrel{\sigma}{\sim} x$.

Proof. By Remark 3.4 and symmetry, we may assume without loss of generality that $V = \{u, v, w, x\}$, and $u \in S(G)$ or $v \in S(G)$. If $u \in S(G)$, then we have $u \stackrel{-\sigma}{\sim} w$ by (S2) since $w \stackrel{-\sigma}{\sim} v \stackrel{\sigma}{\sim} u$, we have $u \stackrel{\sigma}{\sim} x$ by (S2) since $x \stackrel{\sigma}{\sim} w \stackrel{-\sigma}{\sim} u$, and we have $v \stackrel{\sigma}{\sim} x$ by (S1) since $v, x \in N_\sigma(u)$. On the other hand, if $v \in S(G)$, then we have $v \stackrel{\sigma}{\sim} x$ by (S2) since $x \stackrel{\sigma}{\sim} w \stackrel{-\sigma}{\sim} v$, and we have $u \stackrel{\sigma}{\sim} x$ by (S1) since $u, x \in N_\sigma(v)$. Thus the claim holds in any case. \square

5. A full characterization of signed-eliminable graphs

In this section, we state and prove a full characterization of SE graphs, which is the main contribution of this paper.

5.1. The statement

Before giving our characterization, we introduce the following terminology: We call an induced path in G of the form $u \stackrel{\sigma}{\sim} v \stackrel{-\sigma}{\sim} w \stackrel{\sigma}{\sim} x$, where $\sigma \in \{+, -\}$, an *alternating 4-path*. Then our full characterization is described as the following theorem:

Theorem 5.1. Let G be a signed graph. Then G is signed-eliminable if and only if all of the following three conditions are satisfied:

(C1) Both G_+ and G_- are chordal.

- (C2) For any alternating 4-path $u \xrightarrow{\sigma} v \xrightarrow{-\sigma} w \xrightarrow{\sigma} x$ in G (see above for terminology), we have either $w \xrightarrow{-\sigma} u \xrightarrow{\sigma} x$ or $u \xrightarrow{\sigma} x \xrightarrow{-\sigma} v$.
 (C3) G contains no mountain and no hill as an induced subgraph (see Definition 4.4 for terminology).

The “only if” part of Theorem 5.1 follows from Remark 3.3, Lemma 4.6, Remark 3.4 and Lemma 4.5. In the rest of this section, we prove the “if” part; that is, G is SE if the conditions (C1)–(C3) are satisfied.

Remark 5.2. In a previous version of this paper, the characterization was stated in the following form: A signed graph G is signed-eliminable if and only if (C1) and (C3) are satisfied and any induced subgraph of G with four vertices is signed-eliminable. This characterization is also valid by Theorem 5.1, Remark 3.4 and Lemma 4.6. (Note that we do not use this fact in our proof of Theorem 5.1.)

5.2. Some lemmas

This subsection is devoted to present the following lemmas that will be used in our proof of the main theorem:

Lemma 5.3. Suppose that the conditions (C1) and (C2) in Theorem 5.1 are satisfied. If $\sigma \in \{+, -\}$, $k \geq 2$, $x_1 x_2 \cdots x_k$ is an induced path in G_σ , $x_1 \xrightarrow{\sigma} x_0 \neq x_2$ and $x_0 \not\xrightarrow{-\sigma} x_2$, then $x_0 x_1 \cdots x_k$ is also an induced path in G .

Proof. We proceed the proof by induction on k . The case $k = 2$ is trivial, therefore suppose that $k \geq 3$ and $x_0 x_1 \cdots x_{k-1}$ is an induced path in G . Note that $x_0 \neq x_i$ for any $1 \leq i \leq k$ by the assumption. Now if $x_k \xrightarrow{-\sigma} x_i$ for some $1 \leq i \leq k-2$, then we have $x_{i-1} \xrightarrow{\sigma} x_i \xrightarrow{-\sigma} x_k \xrightarrow{\sigma} x_{k-1}$ and $x_{i-1} \not\xrightarrow{\sigma} x_{k-1}$, contradicting (C2). If $x_k \xrightarrow{-\sigma} x_0$, then we have $x_{k-1} \xrightarrow{\sigma} x_k \xrightarrow{-\sigma} x_0 \xrightarrow{\sigma} x_1$, while $x_{k-1} \not\xrightarrow{\sigma} x_0$ and $x_k \not\xrightarrow{\sigma} x_1$ by the assumption and the induction hypothesis. This contradicts (C2). Moreover, if $x_k \xrightarrow{\sigma} x_0$, then $x_0 x_1 \cdots x_k x_0$ is a cycle in G_σ with at least four vertices, while this cycle has no chord since both $x_0 x_1 \cdots x_{k-1}$ and $x_1 x_2 \cdots x_k$ are induced paths in G_σ by the assumption and the induction hypothesis. This contradicts (C1). Hence $x_0 x_1 \cdots x_k$ is also an induced path in G , concluding the proof. \square

Here we introduce the following notations. For subsets $V' \subset V''$ of V and $\sigma \in \{+, -\}$, we define $\text{cl}_\sigma(V'; V'')$ to be the union of vertex sets of the connected components of $G_\sigma|_{V''}$ that have nonempty intersection with V' , and define

$$\begin{aligned}\overline{\text{cl}}_\sigma(V'; V'') &= \text{cl}_\sigma(V'; V'') \cup \{v \in V'' \mid N_{-\sigma}(v) \cap \text{cl}_\sigma(V'; V'') \neq \emptyset\}, \\ \partial_\sigma(V'; V'') &= \overline{\text{cl}}_\sigma(V'; V'') \setminus \text{cl}_\sigma(V'; V'').\end{aligned}$$

Lemma 5.4. Let $V' \subset V'' \subset V$, $\sigma \in \{+, -\}$ and put $W = \text{cl}_\sigma(V'; V'')$ and $\overline{W} = \overline{\text{cl}}_\sigma(V'; V'')$. Suppose that the condition (C2) is satisfied and every connected component of $G_\sigma|_W$ contains at least two vertices. Then $S(G|_{\overline{W}}) \subset W$ and $S(G|_{\overline{W}}) \subset S(G|_{V''})$.

Proof. Let $v \in S(G|_{\overline{W}})$. First, to prove that $v \in W$, it suffices to consider the case that $N_{-\sigma}(v) \cap W \neq \emptyset$. Then we have $v \xrightarrow{-\sigma} u$ for some $u \in W$, while $u \xrightarrow{\sigma} w$ for some $w \in W$ by the assumption. Now condition (S2) implies that $v \xrightarrow{\sigma} w$, therefore $v \in W$ since $v \in V''$ and $w \in W$.

From now, we show that $v \in S(G|_{V''})$. Put $G' = G|_{\overline{W}}$ and $G'' = G|_{V''}$. Now since $v \in W$, the definition of \overline{W} implies that $N_{G''}^+[v] \subset W$ and $N_{G''}^-[v] \subset \overline{W}$, therefore condition (S1) holds since $v \in S(G')$. Similarly, if $u, w \in V''$ and $u \xrightarrow{-\sigma} w \xrightarrow{\sigma} v$, then $w \in W$ and $u \in \overline{W}$, therefore $u \xrightarrow{-\sigma} v$ by the condition (S2) for v and G' . Finally, suppose that $u, w \in V''$ and $u \xrightarrow{\sigma} w \xrightarrow{-\sigma} v$. Then we have $w \in \overline{W}$ as above, while by the assumption, we have $v \xrightarrow{\sigma} x$ for some $x \in W$. Now we have $u \xrightarrow{\sigma} v$ if $x = u$; thus suppose that $x \neq u$. If $w \in W$, then we have $u \in W$ as above, therefore $u \xrightarrow{\sigma} v$ by the condition (S2) for v and G' . On the other hand, if $w \in \overline{W} \setminus W$, then we have $w \xrightarrow{\sigma} x$, therefore (C2) implies that $x \xrightarrow{\sigma} u \xrightarrow{\sigma} v$ (since $u \xrightarrow{\sigma} w \xrightarrow{-\sigma} v \xrightarrow{\sigma} x$ is an alternating 4-path). Thus $u \xrightarrow{\sigma} v$ in any case, therefore condition (S2) holds. Hence the proof is concluded. \square

Lemma 5.5. Let $G = (V, E)$ be a connected signed-eliminable graph such that $E_+ \neq \emptyset$ and $E_- \neq \emptyset$. Then there exists a vertex $v \in S(G)$ such that $N_+(v) \neq \emptyset$ and $N_-(v) \neq \emptyset$.

Proof. Note that conditions (C1)–(C3) hold by the “only if” part of Theorem 5.1 (that has been proved in Section 5.1). By Lemma 2.4, we have $u \xrightarrow{+} v \xrightarrow{-} w$ for some vertices u, v and w of G . Now put $W = \text{cl}_+(v; V)$ and $\overline{W} = \overline{\text{cl}}_+(v; V)$. Then $(G|_W)_+$ is connected and contains u and v , therefore $S(G|_{\overline{W}}) \subset W$ and $S(G|_{\overline{W}}) \subset S(G)$ by Lemma 5.4. On the other hand, now $G|_{\overline{W}}$ is connected and contains w . Thus by putting $X = \text{cl}_-(v; \overline{W})$ and $\overline{X} = \overline{\text{cl}}_-(v; \overline{W})$, it follows that $(G|_{\overline{X}})_-$ is connected and contains v and w , therefore $S(G|_{\overline{X}}) \subset X$ and $S(G|_{\overline{X}}) \subset S(G|_{\overline{W}})$ by Lemma 5.4. Moreover, a vertex $x \in S(G|_{\overline{X}})$ exists by Remark 3.4. Summarizing, we have $x \in S(G)$, $x \in W \cap X$, and both $(G|_W)_+$ and $(G|_X)_-$ are connected and contain at least two vertices, therefore $N_+(x) \neq \emptyset$ and $N_-(x) \neq \emptyset$. Thus the proof is concluded. \square

5.3. Proof of Theorem 5.1, first step

In Sections 5.3–5.5, we give a proof of the “if” part of Theorem 5.1, namely we show that any signed graph $G = (V, E)$ satisfying the conditions (C1)–(C3) is signed-eliminable. Since the conditions (C1)–(C3) are closed under taking induced

subgraphs, we proceed the proof by induction on $|V|$. By [Proposition 4.1](#), the claim is trivial if $|V| \leq 3$; thus suppose that $|V| \geq 4$. Moreover, owing to [Remarks 3.3](#) and [3.5](#), the claim follows if either G is not connected, or $E_+ = \emptyset$ or $E_- = \emptyset$. Thus we may assume further that G is connected, $E_+ \neq \emptyset$ and $E_- \neq \emptyset$, therefore we have $v \stackrel{+}{\sim} v' \stackrel{-}{\sim} v''$ for some vertices v, v' and v'' of G by [Lemma 2.4](#). Now by [Lemma 3.10](#), it suffices to show that $S(G) \neq \emptyset$.

In this subsection, we consider the case that

$$\text{if } u, u', u'' \in V \text{ and } u \stackrel{+}{\sim} u' \stackrel{-}{\sim} u'', \text{ then } u \sim u'', \quad (2)$$

and prove that $S(G) \neq \emptyset$ if condition (2) is satisfied.

Lemma 5.6. *In the above setting, there exists a vertex $w \in V$ such that either $N_{G_+}(w) = \emptyset$ or $N_{G_-}(w) = \emptyset$.*

Proof. Take a pair of a sequence (w_1, w_2, \dots, w_k) of vertices of G and a sequence $(\sigma_2, \sigma_3, \dots, \sigma_k)$ of signs $\sigma_i \in \{+, -\}$, with k maximal, such that $\sigma_i = -\sigma_{i-1}$ for any $3 \leq i \leq k$ and $w_i w_j \in E_{\sigma_j}$ for any $1 \leq i < j \leq k$. Note that $k \geq 3$, since the condition (2) implies that either $v \stackrel{+}{\sim} v''$ (now the pair of (v'', v', v) and $(-, +)$ satisfies the condition) or $v \stackrel{-}{\sim} v''$ (now the pair of (v, v', v'') and $(+, -)$ satisfies the condition). We show that $N_{-\sigma_k}(w_k) = \emptyset$. Assume contrary that $w_k x \in E_{-\sigma_k}$ for some $x \in V$. Note that $x \neq w_i$ for any $1 \leq i \leq k$. Now for each $1 \leq i \leq k-2$, we have $x w_k \in E_{-\sigma_k}$, $w_k w_{k-1} \in E_{\sigma_k}$ and $w_{k-1} w_i \in E_{-\sigma_k}$ (note that $\sigma_k = -\sigma_{k-1}$), therefore $x w_i \in E_{-\sigma_k}$ by condition (C2). Moreover, we have $x w_k \in E_{-\sigma_k}$, $w_k w_{k-2} \in E_{\sigma_k}$ and $w_{k-2} w_{k-1} \in E_{-\sigma_k}$, therefore $x w_{k-1} \in E_{-\sigma_k}$ by condition (C2). Thus we have $x w_i \in E_{-\sigma_k}$ for any $1 \leq i \leq k$, therefore the pair of (w_1, \dots, w_k, x) and $(\sigma_2, \dots, \sigma_k, -\sigma_k)$ also satisfies the condition. This contradicts the maximality of k . Hence we have $N_{-\sigma_k}(w_k) = \emptyset$, therefore the claim holds. \square

Owing to [Lemma 5.6](#), we have $N_{G_\sigma}(w) = \emptyset$ for some $w \in V$ and $\sigma \in \{+, -\}$. Since $E_\sigma \neq \emptyset$, we have $w' \stackrel{\sigma}{\sim} w''$ for some $w', w'' \in V \setminus w$. Now put $W = \text{cl}_\sigma(\{w', w''\}; V \setminus w)$ and $\overline{W} = \overline{\text{cl}}_\sigma(\{w', w''\}; V \setminus w)$. Then $(G|_W)_\sigma$ is connected and contains w' and w'' , while $S(G|_{\overline{W}}) \neq \emptyset$ by the induction hypothesis. Thus [Lemma 5.4](#) implies that $x \in S(G \setminus w)$ for some $x \in W$.

We show that $x \in S(G)$ by using [Lemma 4.3](#), where x and w play the roles of v and v' in that lemma, respectively. If $x \not\sim w$, then condition (D'') follows from condition (2). On the other hand, if $x \stackrel{-\sigma}{\sim} w$, then condition (D'2) (where $-\sigma$ plays the role of σ) holds since $N_\sigma(w) = \emptyset$. For condition (D'1), suppose that $x \stackrel{-\sigma}{\sim} y \neq w$. Then, since $(G|_W)_\sigma$ is connected and contains at least two vertices, we have $x \stackrel{\sigma}{\sim} z$ for some $z \in W$. Now since $N_\sigma(w) = \emptyset$ and $w \stackrel{-\sigma}{\sim} x \stackrel{\sigma}{\sim} z$, we have $w \stackrel{-\sigma}{\sim} z$ by condition (2). Moreover, since $w \stackrel{-\sigma}{\sim} z \stackrel{\sigma}{\sim} x \stackrel{-\sigma}{\sim} y$, we have $w \stackrel{-\sigma}{\sim} y$ by (C2). Thus condition (D'1) is also satisfied. Hence we have $x \in S(G)$ by [Lemma 4.3](#), as desired.

5.4. Proof of Theorem 5.1, second step

From now, we consider the case that condition (2) does not hold, thus we have $v_+ \not\sim v_-$ and $W = N_+(v_+) \cap N_-(v_-) \neq \emptyset$ for some vertices v_+ and v_- of G . Now for each $\sigma \in \{+, -\}$, put

$$\begin{aligned} X_\sigma &= \text{cl}_{-\sigma}(W \cup v_{-\sigma}; W \cup (V \setminus N(v_{-\sigma}))) \setminus (W \cup v_{-\sigma}), \\ Y_\sigma &= \partial_{-\sigma}(W \cup v_{-\sigma}; W \cup (V \setminus N(v_{-\sigma}))), \end{aligned}$$

and put

$$V' = W \cup \{v_+, v_-\} \cup X_+ \cup Y_+ \cup X_- \cup Y_-.$$

By the construction, $(G|_{X_\sigma \cup W \cup v_{-\sigma}})_{-\sigma}$ is connected and contains $W \cup v_{-\sigma}$ for each $\sigma \in \{+, -\}$. Thus [Remark 3.9](#) and [Lemma 5.4](#) imply that

$$S(G|_{V'}) \subset S(G|_{W \cup v_- \cup X_+ \cup Y_+}) \cup S(G|_{W \cup v_+ \cup X_- \cup Y_-}) \subset W \cup \{v_+, v_-\} \cup X_+ \cup X_-,$$

while $v_+, v_- \notin S(G|_{V'})$ by the choice of v_+ and v_- , therefore

$$S(G|_{V'}) \subset W \cup X_+ \cup X_-. \quad (3)$$

On the other hand, the construction implies that, for each $\sigma \in \{+, -\}$,

$$[X_\sigma \cup W, Y_\sigma] \cap E_{-\sigma} = \emptyset \text{ and } (X_\sigma \cup Y_\sigma) \cap N(v_{-\sigma}) = \emptyset. \quad (4)$$

Moreover, we have the following results:

Lemma 5.7. *In the above setting, $X_\sigma \cup Y_\sigma \subset N_\sigma(v_\sigma)$ for each $\sigma \in \{+, -\}$.*

Proof. First, we show that $u \not\stackrel{-\sigma}{\sim} v_\sigma$ for any $u \in X_\sigma \cup Y_\sigma$. If $u \stackrel{-\sigma}{\sim} v_\sigma$, then we have $u \stackrel{-\sigma}{\sim} v_\sigma \stackrel{\sigma}{\sim} w \stackrel{-\sigma}{\sim} v_{-\sigma}$ for any $w \in W$, therefore $u \stackrel{-\sigma}{\sim} v_{-\sigma}$ by (C2). This contradicts (4). Thus we have $u \not\stackrel{-\sigma}{\sim} v_\sigma$.

Now it suffices to show that $u \dashv v_\sigma$ for any $u \in X_\sigma \cup Y_\sigma$. First, for the case $u \in X_\sigma$, we take an induced path $u_0 u_1 \cdots u_k$ in $G_{-\sigma}$ such that $u_0 \in W$, $u_i \in X_\sigma$ for $1 \leq i \leq k-1$ and $u_k = u$ (such a path exists by the construction of X_σ and (4)), and prove that $v_\sigma \dashv u$ by induction on k . We have $v_\sigma \stackrel{\sigma}{\dashv} u_i$ for $0 \leq i \leq k-1$ by induction hypothesis and the previous paragraph. Now by (4) and Lemma 5.3, the path $v_{-\sigma} u_0 u_1 \cdots u_k$ in $G_{-\sigma}$ is an induced path in G , therefore $(v_{-\sigma}, u_0, \dots, u_k; v_\sigma)$ is a σ -mountain if $v_\sigma \not\vdash u$. Thus (C3) implies that $v_\sigma \dashv u$. Secondly, for the case $u \in Y_\sigma$, by the construction and (4), we have $u \stackrel{\sigma}{\dashv} w$ for some $w \in W \cup X_\sigma$. Now if $w \in X_\sigma$, then $w \stackrel{-\sigma}{\dashv} x$ for some $x \in W \cup X_\sigma$, therefore $v_\sigma \stackrel{\sigma}{\dashv} x \stackrel{-\sigma}{\dashv} w \stackrel{\sigma}{\dashv} u$ by the previous paragraph. Thus (C2) implies that $v_\sigma \stackrel{\sigma}{\dashv} u$. On the other hand, suppose that $w \in W$. Then (4) implies that $(v_\sigma, w, u; v_{-\sigma})$ is a $(-\sigma)$ -mountain if $v_\sigma \not\vdash u$, contradicting (C3). Thus we have $v_\sigma \dashv u$. Hence the proof is concluded. \square

Lemma 5.8. *In the above setting, if $\sigma \in \{+, -\}$ and $u \in N_{-\sigma}(v_\sigma)$, then $N_{-\sigma}(v_\sigma) \cup W \cup v_{-\sigma} \cup X_\sigma \cup X_{-\sigma} \cup Y_{-\sigma} \subset N_{-\sigma}[u]$.*

Proof. First, suppose that $w \in W$. Then $u \stackrel{-\sigma}{\dashv} v_\sigma \stackrel{\sigma}{\dashv} w \stackrel{-\sigma}{\dashv} v_{-\sigma}$, therefore $v_{-\sigma} \stackrel{-\sigma}{\dashv} u \stackrel{-\sigma}{\dashv} w$ by (C2) since $v_\sigma \not\vdash v_{-\sigma}$. Moreover, if $u \neq u' \in N_{-\sigma}(v_\sigma)$, then we have $u' \stackrel{-\sigma}{\dashv} v_\sigma \stackrel{\sigma}{\dashv} w \stackrel{-\sigma}{\dashv} u$, therefore $u' \stackrel{-\sigma}{\dashv} u$ by (C2). These imply that $v_{-\sigma} \cup W \cup N_{-\sigma}(v_\sigma) \subset N_{-\sigma}[u]$. On the other hand, if $x \in X_\sigma$, then we have $x' \stackrel{-\sigma}{\dashv} x$ for some $x' \in W \cup X_\sigma$. Now by Lemma 5.7, we have $x \stackrel{-\sigma}{\dashv} x' \stackrel{\sigma}{\dashv} v_\sigma \stackrel{-\sigma}{\dashv} u$, therefore $x \stackrel{-\sigma}{\dashv} u$ by (C2). Thus we have $X_\sigma \subset N_{-\sigma}[u]$.

From now, we show that $x \dashv u$ for any $x \in X_{-\sigma} \cup Y_{-\sigma}$. First, if $x \dashv u$, then $v_\sigma \dashv u \stackrel{\sigma}{\dashv} x \stackrel{-\sigma}{\dashv} v_{-\sigma}$ by Lemma 5.7, while $v_\sigma \not\vdash v_{-\sigma}$. This contradicts (C2), therefore we have $x \not\vdash u$. Secondly, in the case $x \in X_{-\sigma}$, we take an induced path $x_0 x_1 \cdots x_k$ in G_σ such that $x_0 \in W$, $x_i \in X_{-\sigma}$ for $1 \leq i \leq k-1$ and $x_k = x$ (it exists by construction of $X_{-\sigma}$ and (4)), and we prove $x \dashv u$ by induction on k . We have $x_i \stackrel{-\sigma}{\dashv} u$ by induction hypothesis, while the path $v_\sigma x_0 x_1 \cdots x_k$ in G_σ is an induced path in G by (4) and Lemma 5.3. Now if $x \not\vdash u$, then $x \not\vdash u$ by the above result, therefore $(v_\sigma, x_0, \dots, x_k; u, v_{-\sigma})$ is a $(-\sigma)$ -hill. This contradicts (C3), therefore we have $x \dashv u$. Finally, in the case $x \in Y_{-\sigma}$, we have $x \dashv x'$ for some $x' \in X_{-\sigma} \cup W$, and $x' \stackrel{\sigma}{\dashv} x''$ for some $x'' \in v_\sigma X_{-\sigma} \cup W \cup v_\sigma$. Now we have $x \dashv x' \stackrel{\sigma}{\dashv} x'' \stackrel{-\sigma}{\dashv} u$ by the above result, therefore $x \dashv u$ by (C2). Hence we have $X_{-\sigma} \cup Y_{-\sigma} \subset N_{-\sigma}[u]$, therefore the proof is concluded. \square

Lemma 5.9. *In the above setting, if $w \in W$, then $N_\sigma(w) \subset W \cup v_\sigma \cup X_\sigma \cup Y_\sigma \cup X_{-\sigma} \cup N_\sigma(v_{-\sigma})$ for each $\sigma \in \{+, -\}$.*

Proof. Let $u \in N_\sigma(w)$. It suffices to consider the case that $u \neq v_\sigma$ and $u \not\vdash v_{-\sigma}$. Note that $u \not\vdash v_\sigma$ by Lemma 5.8. Now we have $u \in X_{-\sigma}$ if $u \not\vdash v_\sigma$, while $u \in X_\sigma \cup Y_\sigma$ if $u \not\vdash v_{-\sigma}$. Moreover, we have $u \in W$ if $u \stackrel{\sigma}{\dashv} v_\sigma$ and $u \stackrel{-\sigma}{\dashv} v_{-\sigma}$. Hence the claim holds in any case. \square

Lemma 5.10. *In the above setting, if $\sigma \in \{+, -\}$ and $x \in X_\sigma$, then $N_\sigma(x) \subset W \cup v_\sigma \cup X_\sigma \cup Y_\sigma \cup N_\sigma(v_{-\sigma})$ and $N_{-\sigma}(x) \subset W \cup X_\sigma \cup X_{-\sigma} \cup Y_{-\sigma} \cup N_{-\sigma}(v_\sigma)$.*

Proof. First, we show that $u \in W \cup v_\sigma \cup X_\sigma \cup Y_\sigma \cup N_\sigma(v_{-\sigma})$ for any $u \in N_\sigma(x)$. It suffices to consider the case that $u \neq v_\sigma$ and $u \not\vdash v_{-\sigma}$. Now by the choice of x , we have $x \dashv x'$ for some $x' \in X_\sigma \cup W$, therefore $u \stackrel{\sigma}{\dashv} x \dashv x' \stackrel{\sigma}{\dashv} v_\sigma$ by Lemma 5.7. Thus $u \stackrel{\sigma}{\dashv} v_\sigma$ by (C2). This implies that $u \in W$ if $u \dashv v_{-\sigma}$, while $u \in X_\sigma \cup Y_\sigma$ if $u \not\vdash v_{-\sigma}$ (since $u \stackrel{\sigma}{\dashv} x \in X_\sigma$). Thus the claim for $N_\sigma(x)$ holds (since $u \neq v_{-\sigma}$).

Secondly, we show that $u \in W \cup X_\sigma \cup X_{-\sigma} \cup Y_{-\sigma} \cup N_{-\sigma}(v_\sigma)$ for any $u \in N_{-\sigma}(x)$. Note that $u \neq v_\sigma$ and $u \not\vdash v_{-\sigma}$ by (4) and Lemma 5.7, and $u \not\vdash v_{-\sigma}$ by Lemma 5.8 (since $x \not\vdash u$). It suffices to consider the case that $u \not\vdash v_\sigma$. Now we have $u \in X_\sigma$ if $u \not\vdash v_{-\sigma}$, while $u \in W$ if $u \stackrel{\sigma}{\dashv} v_\sigma$ and $u \stackrel{-\sigma}{\dashv} v_{-\sigma}$. Finally, suppose that $u \not\vdash v_\sigma$ and $u \stackrel{-\sigma}{\dashv} v_{-\sigma}$. Take an induced path $x_0 x_1 \cdots x_k$ in $G_{-\sigma}$ such that $x_0 \in W$, $x_i \in X_\sigma$ for $1 \leq i \leq k-1$ and $x_k = x$ (it exists by construction of X_σ and (4)). Then (4) and Lemma 5.3 imply that the path $v_{-\sigma} x_0 x_1 \cdots x_k$ is an induced path in $G_{-\sigma}$, while $v_{-\sigma} \dashv u \stackrel{-\sigma}{\dashv} x_k$. Thus (C1) implies that $u \dashv x_i$ for any $0 \leq i \leq k-1$; in particular $v_\sigma \not\vdash u \dashv x_0 \in W$, therefore $u \in X_{-\sigma} \cup Y_{-\sigma}$. Hence the claim for $N_{-\sigma}(x)$ holds, therefore the proof is concluded. \square

Lemma 5.11. *In the above setting, if $\sigma \in \{+, -\}$ and $y \in Y_\sigma$, then $N_{-\sigma}(y) \subset V' \cup N_{-\sigma}(v_\sigma)$.*

Proof. Let $u \in N_{-\sigma}(y)$. Then $u \neq v_\sigma$ and $u \not\vdash v_{-\sigma}$ by Lemma 5.7 and (4). Now we have $y \stackrel{\sigma}{\dashv} x$ for some $x \in X_\sigma \cup W$, and $x \dashv w$ for some $w \in X_\sigma \cup W \cup v_{-\sigma}$. Thus $u \dashv y \stackrel{\sigma}{\dashv} x \dashv w$, therefore (C2) implies that $u \dashv x$ (since $y \not\vdash w$ by (4)). Now since $x \in X_\sigma \cup W$, the claim follows from Lemmas 5.9 and 5.10. \square

Owing to these results, here we prove that $S(G) \neq \emptyset$ in the case $V' \neq V$. By the induction hypothesis, there is a vertex $v \in S(G|_{V'})$. By (3), we have $v \in W \cup X_\sigma$ for some $\sigma \in \{+, -\}$. From now, we show that $v \in S(G)$.

For the condition (S1), we show that $u \stackrel{\tau}{\dashv} u'$ if $\tau \in \{+, -\}$, $u, u' \in N_\tau(v)$ and $u \neq u'$. This holds if $u, u' \in V'$ since $v \in S(G|_{V'})$; thus suppose that $u \notin V'$ or $u' \notin V'$, say $u \notin V'$. Then, since $v \in W \cup X_\sigma$, we have $u \in N_\tau(v_{-\tau})$ by Lemmas 5.9 and 5.10. Moreover, we have $N_\tau(v) \subset N_\tau[u]$ by Lemmas 5.8–5.10. Thus we have $u \stackrel{\tau}{\dashv} u'$, as desired. Hence the condition (S1) is satisfied.

For the condition (S2), first we show that $v \stackrel{-\sigma}{\sim} u$ if $u \stackrel{-\sigma}{\sim} u' \stackrel{\sigma}{\sim} v$. This holds if $u, u' \in V'$ since $v \in S(G|_{V'})$; thus suppose that $u \notin V'$ or $u' \notin V'$. Now if $u' \in V'$ and $u \notin V'$, then we have $u' \in V' \setminus (Y_{-\sigma} \cup v_{-\sigma})$ by Lemmas 5.9 and 5.10, therefore $u \in N_{-\sigma}(v_{-\sigma})$ by Lemmas 5.9–5.11. Thus Lemma 5.8 implies that $v \stackrel{-\sigma}{\sim} u$ in this case. On the other hand, if $u' \notin V'$, then $u' \in N_{\sigma}(v_{-\sigma})$ by Lemmas 5.9 and 5.10. Now since $v \in X_{\sigma} \cup W$, we have $v \stackrel{-\sigma}{\sim} x$ for some $x \in X_{\sigma} \cup W \cup v_{-\sigma}$, while $x \stackrel{\sigma}{\sim} u'$ by Lemma 5.8. Thus we have $v \stackrel{-\sigma}{\sim} x \stackrel{\sigma}{\sim} u' \stackrel{-\sigma}{\sim} u$, therefore (C2) implies that $v \stackrel{-\sigma}{\sim} u$ in this case. Hence we have $v \stackrel{-\sigma}{\sim} u$, as desired.

Secondly, we show that $v \stackrel{\sigma}{\sim} u$ if $u \stackrel{\sigma}{\sim} u' \stackrel{-\sigma}{\sim} v$. This holds if $u, u' \in V'$ since $v \in S(G|_{V'})$; thus suppose that $u \notin V'$ or $u' \notin V'$. Now if $u' \in V'$ and $u \notin V'$, then we have $u' \in V' \setminus (Y_{\sigma} \cup v_{\sigma})$ by Lemmas 5.9 and 5.10, therefore $u \in N_{\sigma}(v_{-\sigma})$ by Lemmas 5.9–5.11. Thus Lemma 5.8 implies that $v \stackrel{\sigma}{\sim} u$ in this case. On the other hand, if $u' \notin V'$, then $u' \in N_{-\sigma}(v_{\sigma})$ by Lemmas 5.9 and 5.10. Now we have $u \stackrel{\sigma}{\sim} u' \stackrel{-\sigma}{\sim} v_{\sigma} \stackrel{\sigma}{\sim} v$ by Lemma 5.7, therefore (C2) implies that $v \stackrel{\sigma}{\sim} u$ in this case. Thus we have $v \stackrel{\sigma}{\sim} u$, as desired. Hence the condition (S2) holds.

Thus we have $S(G) \neq \emptyset$ if $V' \neq V$, as desired.

Moreover, by applying the above result, we have the following corollaries:

Corollary 5.12. *In the above setting, suppose that $u \stackrel{\sigma}{\sim} v \stackrel{-\sigma}{\sim} w \stackrel{\sigma}{\sim} x$ and $u \not\sim w$ for some $\sigma \in \{+, -\}$ and distinct vertices u, v, w , and x . Then $S(G) \neq \emptyset$.*

Proof. By (C2), we have $u \stackrel{\sigma}{\sim} x$. Now we apply the above argument, where u and w play the roles of v_{σ} and $v_{-\sigma}$, respectively. Then we have $x \notin V'$, since $x \notin W \cup (V \setminus N(v_{\tau}))$ for each $\tau \in \{+, -\}$. Hence we have $S(G) \neq \emptyset$ by the above result, as desired. \square

Corollary 5.13. *In the above setting, suppose that $u \stackrel{+}{\sim} v \stackrel{-}{\sim} w$, $u \not\sim w$ and $N_+[u] \cup N_-[w] \neq V$. Then $S(G) \neq \emptyset$.*

Proof. We apply the above argument, where u and w play the roles of v_+ and v_- , respectively. In this setting, Lemma 5.7 implies that $V' \subset N_+[u] \cup N_-[w]$, therefore $V' \neq V$ by the assumption. Thus we have $S(G) \neq \emptyset$ by the above result. \square

5.5. Proof of Theorem 5.1, final step

Now it suffices to consider the case that condition (2) does not hold and we have $V' = V$ under the notations used in Section 5.4. Moreover, owing to Corollaries 5.12 and 5.13, we may assume without loss of generality that

$$\text{if } u \stackrel{+}{\sim} u' \stackrel{-}{\sim} u'' \text{ and } u \not\sim u'', \text{ then } N_-(u) = N_+(u'') = \emptyset, \quad (5)$$

$$\text{if } u \stackrel{+}{\sim} u' \stackrel{-}{\sim} u'' \text{ and } u \not\sim u'', \text{ then } N_+[u] \cup N_-[u''] = V. \quad (6)$$

By these assumptions, we have $N_{-\sigma}(v_{\sigma}) = \emptyset$ for each $\sigma \in \{+, -\}$. Moreover, we have the following results:

Lemma 5.14. *In the above setting, if $\sigma \in \{+, -\}$ and $y \in Y_{\sigma}$, then we have $N_{-\sigma}(y) = \emptyset$.*

Proof. Assume contrary that $N_{-\sigma}(y) \neq \emptyset$. Owing to construction of Y_{σ} , take an induced path $x_1 x_2 \cdots x_k$ in $G_{-\sigma}$, $k \geq 1$, with k minimal, such that $x_1 \in W$, $x_i \in X_{\sigma}$ for $2 \leq i \leq k$ and $y \stackrel{\sigma}{\sim} x_k$. Put $x_0 = v_{-\sigma}$. Then we have $y \stackrel{\sigma}{\sim} x_k \stackrel{-\sigma}{\sim} x_{k-1}$ and $N_{-\sigma}(y) \neq \emptyset$, while $y \not\sim x_{k-1}$ by (4) and minimality of the k . This contradicts the condition (5). Hence the proof is concluded. \square

Lemma 5.15. *In the above setting, we have $[X_+, X_-] \cap E = \emptyset$, and $[X_{\sigma}, W] \cap E_{\sigma} = \emptyset$ for each $\sigma \in \{+, -\}$.*

Proof. It suffices to show that $[X_{\sigma}, W \cup X_{-\sigma}] \cap E_{\sigma} = \emptyset$ for each $\sigma \in \{+, -\}$. Assume contrary that $x \stackrel{\sigma}{\sim} x'$ for some $x \in X_{\sigma}$ and $x' \in W \cup X_{-\sigma}$. Then we have $x \stackrel{\sigma}{\sim} x' \stackrel{-\sigma}{\sim} v_{-\sigma}$ by Lemma 5.7, while $x \not\sim v_{-\sigma}$ by (4). Moreover, we have $N_{-\sigma}(x) \neq \emptyset$ by construction of X_{σ} . This contradicts (5). Hence the claim holds. \square

Lemma 5.16. *In the above setting, for each $\sigma \in \{+, -\}$, let W_{σ} denote the union of vertex sets of the connected components of $G|_W$ that are joined with X_{σ} by an edge in $E_{-\sigma}$. Then $W_+ \cap W_- = \emptyset$.*

Proof. Assume contrary that $G|_{W_+}$ and $G|_{W_-}$ involve a common connected component. Then there are a vertex $w_0 \in X_+$, an induced path $w_1 w_2 \cdots w_{k-1}$ in $G|_W$ and a vertex $w_k \in X_-$ such that $w_0 \stackrel{-}{\sim} w_1$ and $w_{k-1} \stackrel{+}{\sim} w_k$. Now there exists an index $1 \leq i \leq k-1$ such that $w_{i-1} \stackrel{-}{\sim} w_i \stackrel{+}{\sim} w_{i+1}$. We have $v_+ \stackrel{+}{\sim} w_{i-1}$ and $v_- \stackrel{-}{\sim} w_{i+1}$ by Lemma 5.7, therefore $v_+ \stackrel{+}{\sim} w_{i+1}$ and $v_- \stackrel{-}{\sim} w_{i-1}$ by (C2). Thus we have $w_{i-1}, w_{i+1} \in W$, therefore $2 \leq i \leq k-2$ and $w_{i-1} \not\sim w_{i+1}$ since $w_1 \cdots w_{k-1}$ is an induced path in G . This contradicts (5). Hence the proof is concluded. \square

Lemma 5.17. *In the above setting, if $v \in V$, $N_+(v) \neq \emptyset$ and $N_-(v) \neq \emptyset$, then the condition (S1) for v implies the condition (S2) for v .*

Proof. Suppose that (S1) holds, and let $\sigma \in \{+, -\}$ and $u \stackrel{\sigma}{\sim} w \stackrel{\sigma}{\sim} v$. Then we have $N_{-\sigma}(v) \neq \emptyset$ by the assumption, therefore $u \sim v$ by (5). Moreover, if $u \stackrel{\sigma}{\sim} v$, then $u, w \in N_{\sigma}(v)$ and $u \not\stackrel{\sigma}{\sim} w$, contradicting the condition (S1). Hence we have $u \stackrel{\sigma}{\sim} v$, therefore the condition (S2) holds. \square

Now suppose that $X_+ \neq \emptyset$ and $X_- \neq \emptyset$, therefore $W_+ \neq \emptyset$ and $W_- \neq \emptyset$. Put $V_1 = V \setminus (W_- \cup X_-)$ and $G_1 = G|_{V_1}$. Then G_1 is connected by Lemma 5.7, while $V_1 \neq V$ and G_1 has an edge in E_+ and an edge in E_- . Thus by the induction hypothesis and Lemma 5.5, we have $v \in S(G_1)$, $N_{(G_1)_+}(v) \neq \emptyset$ and $N_{(G_1)_-}(v) \neq \emptyset$ for some $v \in V_1$. Now we have $v \in (W \setminus W_-) \cup X_+$ by Lemma 5.14, therefore $N_G(v) \cap (W_- \cup X_-) = \emptyset$ by Lemma 5.15 and the construction of W_+ and W_- . Thus we have $N_G(v) = N_{G_1}(v)$, therefore condition (S1) for v and G holds since $v \in S(G_1)$. Hence Lemma 5.17 implies that $v \in S(G)$ in this case.

Finally, suppose that $X_+ = \emptyset$ or $X_- = \emptyset$, say $X_- = \emptyset$. Now we have the following property:

Lemma 5.18. *In the above setting, $Y_- \cup v_-$ is a clique in G_- .*

Proof. By Lemma 5.7, it suffices to show that Y_- is a clique in G_- . Let $y \in Y_-$. Then, since $X_- = \emptyset$, we have $y \stackrel{-}{\sim} w$ for some $w \in W$ by construction of Y_- . Thus we have $v_+ \stackrel{+}{\sim} w \stackrel{-}{\sim} y$, while $v_+ \not\sim y$ and $(Y_- \setminus y) \cap N_+(v_+) = \emptyset$ by (4). Therefore condition (6) implies that $Y_- \setminus y \subset N_-(y)$. Hence the claim holds. \square

Put $V_2 = V \setminus v_+$ and $G_2 = G|_{V_2}$. Then we have $S(G_2) \neq \emptyset$ by the induction hypothesis. More strongly, there is a $v \in S(G_2)$ such that $v \in W \cup X_+ \cup Y_+$. In fact, since G_2 is connected (every vertex in G_2 is joined with v_- by a path in G_2), this holds by Lemmas 5.5 and 5.14 if G_2 has an edge in E_+ , and by Lemmas 2.3 and 5.18 if G_2 has no edge in E_+ . Now we have $N_{G_+}(v) \subset V \setminus (Y_- \cup v_-)$ by Lemma 5.14, while $N_-(v_+) = \emptyset$ and $V \setminus (Y_- \cup v_-) \subset N_{G_+}[v_+]$ by Lemma 5.7. Thus the assumption and condition (D) in Lemma 4.2 are satisfied, where $\{v_+\}$ plays the role of V'' , therefore we have $v \in S(G)$ by that lemma.

Hence the proof of the “if” part of Theorem 5.1 is concluded.

6. Special cases

In this section, we apply Theorem 5.1 to characterize the SE graphs in some subclasses. Let $G = (V, E)$ be a signed graph throughout this section. First, we consider the case of signed graphs with four vertices:

Proposition 6.1. *If $|V| = 4$, then G is signed-eliminable if and only if one of the following conditions is satisfied:*

(FV1) G_+ or G_- has a vertex of degree three.

(FV2) Both G_+ and G_- are chordal, G is not a mountain, and G has no alternating 4-path.

Proof. For the “only if” part, suppose that conditions (C1)–(C3) in Theorem 5.1 are satisfied and (FV2) does not hold. Then by (C1) and (C3), the failure of (FV2) implies that G has an alternating 4-path, therefore (FV1) follows from (C2). On the other hand, for the “if” part, suppose that (FV1) or (FV2) holds. Now if (FV2) holds, then all of (C1), (C2) and (C3) follow, since any hill with four vertices involves an alternating 4-path. Alternatively, suppose that (FV1) holds. Then (C1) holds since an induced cycle with four vertices in G has no vertex of degree three, and (C3) holds by the definition of mountains and hills. Moreover, if G has an alternating 4-path $u \stackrel{\sigma}{\sim} v \stackrel{-\sigma}{\sim} w \stackrel{\sigma}{\sim} x$ with $\sigma \in \{+, -\}$, then neither v nor w has degree three in G_+ or G_- , therefore either u or x has degree three in G_{σ} . This implies that $w \stackrel{\sigma}{\sim} u \stackrel{\sigma}{\sim} x$ or $u \stackrel{\sigma}{\sim} x \stackrel{\sigma}{\sim} v$, therefore (C2) holds. Hence the proof is concluded. \square

Note that a list of the non-SE graphs with four vertices is given in [2].

Secondly, we consider the case that the underlying graph G is chordal:

Proposition 6.2. *Suppose that G is chordal (as a non-signed graph). Then G is signed-eliminable if and only if conditions (C2) and (C3) in Theorem 5.1 are satisfied.*

Proof. It suffices to show that both G_+ and G_- are chordal if G is chordal and G satisfies (C2) and (C3). Let $\sigma \in \{+, -\}$ and let $v_0 v_1 \cdots v_{k-1} v_0$ be a cycle in G_{σ} with $k \geq 4$. Then, since G is chordal, we have $v_i \sim v_j$ for some indices i and j with $j \neq i \pm 1$ (where indices are considered in modulo k). The claim holds if $v_i \stackrel{\sigma}{\sim} v_j$, thus suppose that $v_i \stackrel{-\sigma}{\sim} v_j$. Then we have $v_{i-1} \stackrel{\sigma}{\sim} v_i \stackrel{-\sigma}{\sim} v_j \stackrel{\sigma}{\sim} v_{j-1}$, therefore $v_{i-1} \stackrel{\sigma}{\sim} v_{j-1}$ by (C2). Hence the claim holds. \square

Moreover, we consider the case that the underlying graph G has no independent set of size three:

Proposition 6.3. *Suppose that G has no three distinct vertices u, u' and u'' such that $u \not\sim u' \not\sim u'' \not\sim u$ (i.e. the independence number $\alpha(G)$ of G is less than three). Then G is signed-eliminable if and only if condition (C2) in Theorem 5.1 and the following two conditions are satisfied:*

(I1) Both G_+ and G_- has no cycle of length four or five which is an induced cycle in G .

(I2) G contains no hill with five or six vertices as an induced subgraph.

Proof. The “only if” part follows from Theorem 5.1. To prove the “if” part, we show that conditions (C1) and (C3) hold if (C2), (I1) and (I2) are satisfied. For condition (C1), let $v_0 v_1 \cdots v_{k-1} v_0$ be a cycle in G_σ , with $\sigma \in \{+, -\}$ and $k \geq 4$. Now we have $v_i \xrightarrow{\sigma} v_j$ for some indices $i \neq j$ with $j \neq i \pm 1$ (where indices are considered in modulo k). In fact, this follows from (I1) if $k \leq 5$, while this follows if $k \geq 6$ since now $\{v_0, v_2, v_4\}$ does not form an independent set by the assumption. The condition (C1) holds if $v_i \xrightarrow{\sigma} v_j$, thus suppose that $v_i \xrightarrow{-\sigma} v_j$. Now we have $v_{i-1} \xrightarrow{\sigma} v_i \xrightarrow{-\sigma} v_j \xrightarrow{-\sigma} v_{j-1}$, therefore $v_{i-1} \xrightarrow{\sigma} v_{j-1}$ by (C2). Thus the condition (C1) is satisfied.

For condition (C3), note that any mountain $(v_1, v_2, \dots, v_n; w)$ has an independent set $\{v_1, v_n, w\}$, and any hill $(v_1, v_2, \dots, v_n; w_1, w_2)$ with $n \geq 5$ has an independent set $\{v_1, v_3, v_5\}$. Thus by the assumption, these graphs do not appear in G as induced subgraphs. Moreover, (C2) implies that any hill with four vertices does not appear in G as an induced subgraph. Hence the condition (C3) follows from (I2), therefore the proof is concluded. \square

Corollary 6.4. Suppose that G is a complete graph (as a non-signed graph). Then G is signed-eliminable if and only if for each $\sigma \in \{+, -\}$, G_σ contains, as an induced subgraph, neither a simple path with four vertices, nor a pair of two disjoint edges such that no vertex of one edge is joined by an edge in G_σ with a vertex of another edge.

Proof. Since conditions (I1) and (I2) in Proposition 6.3 are always satisfied by the assumption, it suffices to show that (C2) is now equivalent to the condition in the statement. First, if (C2) holds then the condition in the statement is satisfied, since the two kinds of subgraphs in the statement do not satisfy the condition (C2). On the other hand, suppose that the condition in the statement holds and G has an alternating 4-path $u \xrightarrow{\sigma} v \xrightarrow{-\sigma} w \xrightarrow{\sigma} x$. Put $V' = \{u, v, w, x\}$ and $G' = G|_{V'}$. Now if $u \xrightarrow{-\sigma} x$, then the edges uv and wx in G_σ form a pair as in the statement when G'_σ has no more edges; G'_σ is a simple path with four vertices when G'_σ has just one more edge; and the edges ux and vw in $G_{-\sigma}$ form a pair as in the statement when G'_σ has two more edges. Thus the condition in the statement implies that $u \xrightarrow{\sigma} x$. Moreover, since G'_σ is not a simple path with four vertices by the condition, we have either $u \xrightarrow{\sigma} w$ or $v \xrightarrow{\sigma} x$. Thus the condition (C2) holds. Hence the proof is concluded. \square

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